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Modulated Phase of a Potts Model with Competing Binary Interactions on a Cayley Tree

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Abstract We study the phase diagram for Potts model on a Cayley tree with competing nearest-neighbor interactions J_1, prolonged next-nearest-neighbor interactions J_p and one-level next-nearest-neighbor interactions J_o. Vannimenus proved that the phase diagram of Ising model with $J_o = 0$ contains a modulated phase, as found for similar models on periodic lattices, but the multicritical Lifshitz point is at zero temperature. Later Mariz et al. generalized this result for Ising model with $J_o \neq 0$ and recently Ganikhodjaev et al. proved similar result for the three-state Potts model with $J_o = 0$. We consider Potts model with $J_o \neq 0$ and show that for some values of J_o the multicritical Lifshitz point be at non-zero temperature. We also prove that as soon as the same-level interaction J_o is nonzero, the paramagnetic phase found at high temperatures for $J_o = 0$ disappears, while Ising model does not obtain such property. To perform this study, an iterative scheme similar to that appearing in real space renormalization group frameworks is established; it recovers, as particular case, previous work by Ganikhodjaev et al. for $J_o = 0$. At vanishing temperature, the phase diagram is fully determined for all values and signs of J_1, J_p and J_o. At finite temperatures several interesting features are exhibited for typical values of J_o/J_1.

Keywords Potts model · Cayley tree · Phase diagram · Next-nearest-neighbour · Modulated phase

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1 Introduction

Consideration of spin models with multispin interactions has proved to be fruitful in many fields of physics, ranging from the determination of phase diagrams in metallic alloys and exhibition of new types of phase transition, to site percolation.

Systems exhibiting spatially modulated structures, commensurate or incommensurate with the underlying lattice, are of current interest in condensed matter physics [1]. Among the idealized systems for modulated ordering, the axial next-nearest-neighbour Ising (ANNNI) model, originally introduced by Elliot [2] to describe the sinusoidal magnetic structure of Erbium, and the chiral Potts model, introduced by Ostlund [3] and Huse [4] in connection with monolayers adsorbed on rectangular substrates, have been studied extensively by a variety of techniques. A particularly interesting and powerful method is the study of modulated phases through the measure-preserving map generated by the mean-field equations, as applied by Bak [5] and Jensen and Bak [6] to the ANNNI model. The main drawback of the method lies in the fact that thermodynamic solutions correspond to stationary but unstable orbits. However, when these models are defined on Cayley trees, as in the case of the Ising model with competing interactions examined by Vannimenus [7], it turns out that physically interesting solutions correspond to the attractors of the mapping. This simplifies the numerical work considerably, and detailed study of the whole phase diagram becomes feasible. Apart from the intrinsic interest attached to the study of models on trees, it is possible to argue that the results obtained on trees provide a useful guide to the more involved study of their counterparts on crystal lattices.

The ANNNI model, which consists of an Ising model with nearest-neighbour interactions augmented by competing next-nearest-neighbour couplings acting parallel to a single axis direction, is perhaps the simplest nontrivial model displaying a rich phase diagram with a Lifshitz point and many spatially modulated phases. There has been a considerable theoretical effort to obtain the structure of the global phase diagram of the ANNNI model in the $T-p$ space, where T is temperature and $p=-J_2/J_1$ is the ratio between the competing exchange interactions. On the basis of numerical mean-field calculations, Bak and von Boehm [8] suggested the existence of an infinite succession of commensurate phases, the so-called devil's staircase, at low temperatures. This mean-field picture has been supported by low-temperature series expansions performed by Fisher and Selke [9]. At the paramagnetic-modulated boundary analytic mean-field calculations show that the critical wave number varies continuously and vanishes at the Lifshitz point.

A phase diagram of a model describes a morphology of phases, stability of phases, transitions from one phase to another and corresponding transition lines. A Potts model just as an Ising model with competing interactions has recently been studied extensively because of the appearance of nontrivial magnetic orderings (see [7–16] and references therein). The Cayley tree is not a realistic lattice; however, its amazing topology makes the exact calculation of various quantities possible. For many problems the solution on a tree is much simpler than on a regular lattice and is equivalent to the standard Bethe-Peierls theory [17]. On the Cayley tree one can consider two type of next-nearest-neighbours: prolonged and one-level next-nearest-neighbours (definitions see below). In the case of the Ising model with competing nearest-neighbor interactions J and prolonged next-nearest-neighbour interactions J_p Vannimenus [7] was able to find new modulated phases, in addition to the expected paramagnetic and ferromagnetic ones. From this result follows that Ising model with competing interactions on a Cayley tree is real interest since it has many similarities with models on periodic lattices. In fact, it has many common features with them, in particular the existence of a modulated phase, and shows no sign of pathological behavior—at least no more than mean-field theories of similar systems [7]. Moreover a detailed study of its properties was carried out with essentially exact results, using rather simple numerical methods.

This suggest that more complicated models should be studied on trees, with the hope to discover new phases or unusual types of behavior. The important point is that statistical mechanics on trees involve nonlinear recursion equations and are naturally connected to the rich world of dynamical systems, a world presently under intense investigation [7].

In [7] for Ising model on a Cayley tree with competing interactions, where the interactions appear through the parameters $a = \exp(J/T), b = \exp(J_p/T)$, the basic equations have following form:

$$
\begin{aligned}
x' &= \frac{1}{a^2 D}[(1+b^2x)^2 + (y_1 + b^2 y_2)^2],\\
y_1' &= \frac{2}{D}(b^2 y_1 + y_2)(b^2 + x),\\
y_2' &= -\frac{2}{a^2 D}(y_1 + b^2 y_2)(1 + b^2 x)
\end{aligned}
\tag{1}
$$

with

$$
D = (b^2 + x)^2 + (b^2 y_1 + y_2)^2.
$$

Later Mariz et al. [10] extended this results assuming the existence also of an interaction J_o between the one-level nearest-next-neighbours. In this case the basic equations have form:

$$
\begin{aligned}
x' &= (a^2D)^{-1}[b^4(x^2 + y_2^2) + 2(b/c)^2(x + y_1 y_2) + (1 + y_1^2)],\\
y_1' &= 2D^{-1}[b^4 y_1 + (b/c)^2(y_2 + y_1 x) + y_2 x],\\
y_2' &= -2(a^2 D)^{-1}[b^4 y_2 x + (b/c)^2(y_2 + y_1 x) + y_1],
\end{aligned}
\tag{2}
$$

where

$$
D = b^4(1 + y_1^2) + 2(b/c)^2(x + y_1 y_2) + (x^2 + y_2^2)
$$

and $c = \exp(J_o/T)$.

One can see that the presence of one-level next-nearest-neighbours interactions don't complicate the basic equations and respectively considered model. Both the recurrent equations (1) and recurrent equations (2) have a fixed point $(x^*, 0, 0)$ and the transition lines may then be obtained by linearizing the system (1) or (2) around the fixed point $(x^*, 0, 0)$, where x^* is given by

$$
x^* = [(1 + b^2 x^*)/a(b^2 + x^*)]^2
$$

in first case and by

$$
x^* = \frac{b^4 x^{*2} + 2(b/c)^2 x^* + 1}{a^2[b^4 + 2(b/c)^2 x^* + x^{*2}]}
$$

in second case.

The fixed point is linearly stable if the eigenvalues of linearized equations have module smaller than one. Two cases should be examined, according to whether the eigenvalues are real or complex.

In [18, 19] Ising model with competing nearest-neighbour and one-level next-nearest-neighbours interactions with presence of external magnetic field was studied. It is proved that such model is exactly solvable and phase diagram of this model consists of ferromagnetic and antiferromagnetic phases only. Since the phase diagram of Ising model with competing binary interactions contains a modulated phase if and only if nonzero prolonged next-nearest-neighbour interaction is present.

In this paper we consider Potts model with competing binary interactions. The Potts model [20] was introduced as a generalization of the Ising model to more than two components and encompasses a number of problems in statistical physics (see, e.g. [21]) recently. The model is structured richly enough to illustrate almost every conceivable nuance of the subject. In [16], the phase diagram of the three states Potts model with nearest-neighbour interactions J and prolonged next-nearest-neighbors interactions J_p was studied. In this case the basic equations have following form:

$$
\begin{aligned}
x' &= \frac{1}{2bD}[P(y_1, y_2, y_3) + ((a+1)x + 2 - y_1 - ay_2 - y_3)^2],\\
y_1' &= \frac{2}{D}(a+x)(ay_1 + y_2 + y_3),\\
y_2' &= -\frac{1}{bD}[y_1 + ay_2 + y_3][2 + (a+1)x - (a-1)(y_2 - y_3)],\\
y_3' &= \frac{1}{bD}(a-1)(y_3 - y_2)[2 + (a+1)x - 2y_1 - (a+1)(y_2 + y_3)],
\end{aligned}
\tag{3}
$$

where

$$
\begin{aligned}
D &= (a+x)^2 + (ay_1 + y_2 + y_3)^2,\\
P(y_1, y_2, y_3) &= 3y_1^2 + (4a^2 - 4a + 3)y_2^2 + (3a^2 - 4a + 4)y_3^2 + 2(2a+1)y_1y_2\\
&\quad + 2(a+2)y_1y_3 - (2a^2 - 7a + 2)y_2y_3,
\end{aligned}
$$

and $a = \exp(J_p/T), b = \exp(J/T)$.

One can see that this system have a fixed point $(x^*, 0, 0, 0)$ and the transition lines may then be obtained by linearizing it around the fixed point $(x^*, 0, 0, 0)$, where x^* is given by

$$
x^* = \frac{1}{2b}\left[\frac{(1 + ax^*) + 2}{a + x^*}\right]^2.
$$

The diagram of this model consists of five phases: ferromagnetic, paramagnetic, modulated, antiphase and paramodulated, all meet at the point $(T = 0, -J_p/J = 1/3)$. A distinctive feature of the diagram is seen in the existence of a new phase, i.e. paramodulated phase found at low temperatures, forming an island inside the modulated phase. Numerical investigations of the dynamics of (3) show that in some values of the parameters T/J and $-J_p/J$ it has $y_1 = y_2 = y_3 = 0$. In the plane $(T/J, J_p/J)$ such values form a leaf shaped island inside the modulated region (see [16]). In this island, the average magnetization is equal to 0 with respect to the centered set of spins $\{-1, 0, 1\}$ as in the paramagnetic case. Therefore, we denote it as paramodulated phase. It is inherent in the Potts model as no analogue could be found within the Ising setting.

The aim of this paper is to extend the results of [16] to the Potts model with competing nearest-neighbour, prolonged next-nearest-neighbour and one-level next-nearest-neighbour interactions and to clarify the role of one-level next-nearest-neighbour interactions.

The main new result is that the introduction of one-level (same-level) interactions has a strong effect on the phase diagram:

firstly it appears to shift the multicritical Lifshitz to finite temperature, while it was stuck at zero temperature T for all systems with competing interactions, Ising or Potts, studied on the Cayley tree previously [7, 10, 16];

secondary, as soon as the same-level interaction J_o is nonzero, the paramagnetic phase found at high temperatures for $J_o = 0$ disappears, while Ising model does not obtain such property [10].

2 Definitions

Cayley Tree A Cayley tree Γ^k of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles with exactly $k + 1$ edges issuing from each vertex. Let denote the Cayley tree as $\Gamma^k = (V, \Lambda)$, where V is the set of vertices of Γ^k, Λ is the set of edges of Γ^k. Two vertices x and y, $x, y \in V$ are called *nearest-neighbors* if there exists an edge $l \in \Lambda$ connecting them, which is denoted by $l = \langle x, y\rangle$. The distance $d(x, y)$, $x, y \in V$, on the Cayley tree Γ^k, is the number of edges in the shortest path from x to y. For a fixed $x^0 \in V$ we set

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \qquad V_n = \{x \in V \mid d(x, x^0) \leq n\}$$

and L_n denotes the set of edges in V_n. The fixed vertex x^0 is called the 0-th level and the vertices in W_n are called the n-th level. For the sake of simplicity we put $|x| = d(x, x^0)$, $x \in V$. Two vertices $x, y \in V$ are called *the next-nearest-neighbours* if $d(x, y) = 2$. The next-nearest-neighbour vertices x and y are called *prolonged next-nearest-neighbours* if $|x| \neq |y|$ and is denoted by $\widetilde{\rangle x, y\langle}$. The next-nearest-neighbour vertices $x, y \in V$ that are not prolonged are called *one-level next-nearest-neighbours* since $|x| = |y|$ and are denoted by $\overline{\rangle x, y\langle}$.

Below we will consider a semi-infinite Cayley tree Γ^2_+ of order 2, i.e. an infinite graph without cycles with 3 edges issuing from each vertex except for x^0 which has only 2 edges.

The Model For the three-state Potts model with spin values in $\Phi = \{1, 2, 3\}$, the relevant Hamiltonian with competing nearest-neighbour and next-nearest-neighbour binary interactions has the form

$$H(\sigma) = -J_o \sum_{\overline{\rangle x,y\langle}} \delta_{\sigma(x)\sigma(y)} - J_p \sum_{\widetilde{\rangle x,y\langle}} \delta_{\sigma(x)\sigma(y)} - J_1 \sum_{\langle x,y\rangle} \delta_{\sigma(x)\sigma(y)}, \tag{4}$$

where $J_o, J_p, J_1 \in R$ are coupling constants and δ is the Kronecker symbol. This model recovers that in [16] for $J_o = 0$. This choice is not a mere mathematical complication of the previous results since, as we shall see later on, a significantly richer phase diagram is obtained.

Note that the case $J_p = 0$, i.e., the prolonged interaction vanishes, was fully investigated in [15] and [22]. It was produced the following system of recurrent equations:

$$
\begin{aligned}
u_{n+1} &= \frac{c + 2a^2 u_n + 2v_n + 2a^2 u_n v_n + a^4 c u_n^2 + c v_n^2}{a^4 c + c(u_n^2 + v_n^2) + 2u_n v_n + 2a^2(u_n + v_n)}, \\
v_{n+1} &= \frac{c + 2u_n + 2a^2 v_n + 2a^2 u_n v_n + c u_n^2 + a^4 c v_n^2}{a^4 c + c(u_n^2 + v_n^2) + 2u_n v_n + 2a^2(u_n + v_n)}.
\end{aligned}
\tag{5}
$$

Here $a = \exp(J_o/2T)$, $c = \exp(J_1/T)$, $u_n = \frac{Z_2^{(n)}}{Z_1^{(n)}}$, $v_n = \frac{Z_3^{(n)}}{Z_1^{(n)}}$ and $Z_i^{(n)}$ is the partition function on V_n with the spin i in the root x^0, $i = 1, 2, 3$.

Elementary analysis of this system of recurrent equations shows that this restricted model ($J_p = 0$) is exactly solvable, that is, one can find exact value of T^* such that a phase transition occurs for all $T < T^*$, where T^* is a critical value of temperature [22]. In addition the phase diagram of this model consists of ferromagnetic and antiferromagnetic phases only. Thus the phase diagram of Potts model with competing binary interactions like Ising model contains a modulated phase if and only if nonzero prolonged next-nearest-neighbour interaction is present.

3 Basic Equations

Let Γ_+^2 be a Cayley tree of second order, i.e. an infinite graph without cycles with 3 edges issuing from each vertex except for x^0 which has only 2 edges. In order to produce the recurrent equations, we consider the relation of the partition function on V_n to the partition function on subsets of V_{n-1}. Given the initial conditions on V_1, the recurrence equations indicate how their influence propagates down the tree. Below we consider following partition functions:

$Z_i^{(n)}$ is a partition function on V_n with the spin i in the root x^0, $i = 1, 2, 3$;

$Z^{(n)}(i, j)$ is a partition function on V_n with the configuration (i, j) on an edge $\langle x^0, x\rangle$, where $x \in W_1$ and $i, j = 1, 2, 3$;

$Z^{(n)}(i_1, i_0, i_2)$ is a partition function on V_n where the spin in the root x^0 is i_0 and the two spins in the proceeding ones are i_1 and i_2, respectively.

There are 27 different partition functions $Z^{(n)}(i_1, i_0, i_2)$ and the partition function $Z^{(n)}$ in volume V_n can the be written as follows

$$
Z^{(n)} = \sum_{i_1, i_0, i_2 = 1}^{3} Z^{(n)}(i_1, i_0, i_2)
$$

and

$$
Z^{(n)}(i_1, i_0, i_2) = \exp\left(\frac{J_0}{T} \cdot \delta_{i_1 i_2}\right) Z^{(n)}(i_0, i_1) Z^{(n)}(i_0, i_2).
$$

Assume

$$
\begin{aligned}
Z^{(n)}(1,1) &= cA_1^{(n)}, & Z^{(n)}(1,2) &= B_1^{(n)}, & Z^{(n)}(1,3) &= C_1^{(n)}, \\
Z^{(n)}(2,1) &= A_2^{(n)}, & Z^{(n)}(2,2) &= cB_2^{(n)}, & Z^{(n)}(2,3) &= C_2^{(n)}, \\
Z^{(n)}(3,1) &= A_3^{(n)}, & Z^{(n)}(3,2) &= B_3^{(n)}, & Z^{(n)}(3,3) &= cC_3^{(n)}.
\end{aligned}
$$

Then through a direct calculation one gets the following system of recurrent equations:

$$
\begin{aligned}
&Z^{(n)}(1,1,1)=a^2c^2(A_1^{(n)})^2, \qquad Z^{(n)}(1,1,2)=cA_1^{(n)}B_1^{(n)}, \qquad Z^{(n)}(1,1,3)=cA_1^{(n)}C_1^{(n)},\\
&Z^{(n)}(2,1,1)=Z^{(n)}(1,1,2), \qquad Z^{(n)}(2,1,2)=a^2(B_1^{(n)})^2, \qquad Z^{(n)}(2,1,3)=B_1^{(n)}C_1^{(n)},\\
&Z^{(n)}(3,1,1)=Z^{(n)}(1,1,3), \qquad Z^{(n)}(3,1,2)=Z^{(n)}(2,1,3),\\
&Z^{(n)}(3,1,3)=a^2(C_1^{(n)})^2,\\
&Z^{(n)}(1,2,1)=a^2c^2(A_2^{(n)})^2, \qquad Z^{(n)}(1,2,2)=cA_2^{(n)}B_2^{(n)}, \qquad Z^{(n)}(1,2,3)=A_2^{(n)}C_2^{(n)},\\
&Z^{(n)}(2,2,1)=Z^{(n)}(1,2,2), \qquad Z^{(n)}(2,2,2)=a^2c^2(B_2^{(n)})^2,\\
&Z^{(n)}(2,2,3)=cB_2^{(n)}C_2^{(n)},\\
&Z^{(n)}(3,2,1)=Z^{(n)}(1,2,3), \qquad Z^{(n)}(3,2,2)=Z^{(n)}(2,2,3),\\
&Z^{(n)}(3,2,3)=a^2(C_2^{(n)})^2,\\
&Z^{(n)}(1,3,1)=a^2(A_3^{(n)})^2, \qquad Z^{(n)}(1,3,2)=A_3^{(n)}B_3^{(n)}, \qquad Z^{(n)}(1,3,3)=cA_3^{(n)}C_3^{(n)},\\
&Z^{(n)}(2,3,1)=Z^{(n)}(1,3,2), \qquad Z^{(n)}(2,3,2)=a^2(B_3^{(n)})^2, \qquad Z^{(n)}(2,3,3)=cB_3^{(n)}C_3^{(n)},\\
&Z^{(n)}(3,3,1)=Z^{(n)}(1,3,3), \qquad Z^{(n)}(3,3,2)=Z^{(n)}(2,3,3),\\
&Z^{(n)}(3,3,3)=a^2c^2(C_3^{(n)})^2,
\end{aligned}
\tag{6}
$$

where $a=\exp(\frac{J_o}{2T})$; $b=\exp(\frac{J_p}{T})$, $c=\exp(\frac{J_1}{T})$. The asymptotic behavior of this system is defined by the first datum, which is in turn determined by a boundary condition $\bar{\sigma}_n=\{\sigma(x), x\in V\setminus V_n\}$.

For the free boundary we have

$$
\begin{aligned}
&A_1^{(n)}=B_2^{(n)}=C_3^{(n)},\\
&A_2^{(n)}=A_3^{(n)}=B_1^{(n)}=B_3^{(n)}=C_1^{(n)}=C_2^{(n)},
\end{aligned}
$$

so that

$$Z_1^{(n)}=Z_2^{(n)}=Z_3^{(n)}.$$

If consider all partition functions in volume V_n under the boundary condition $\bar{\sigma}_n\equiv 1$, then

$$
\begin{aligned}
&B_1^{(n)}=C_1^{(n)}, \qquad A_2^{(n)}=A_3^{(n)},\\
&B_2^{(n)}=C_3^{(n)}, \qquad B_3^{(n)}=C_2^{(n)},
\end{aligned}
$$

and

$$Z_2^{(n)}=Z_3^{(n)}.$$

Similarly for the boundary condition $\bar{\sigma}_n\equiv 2$ we have

$$Z_1^{(n)}=Z_3^{(n)}$$

and for boundary condition $\bar{\sigma}_n\equiv 3$

$$Z_1^{(n)}=Z_2^{(n)}.$$

where $Z_i^{(n)}$ is the partition function on V_n with the spin i in the root x^0, $i=1,2,3$.

If $J_p = 0$, i.e. $b = 1$, then from the system of equations (6) we derive

$$\begin{aligned} Z_1^{(n+1)} &= a^2c^2(Z_1^{(n)})^2 + 2cZ_1^{(n)}Z_2^{(n)} + 2cZ_1^{(n)}Z_3^{(n)} + a^2(Z_2^{(n)})^2 + 2cZ_2^{(n)}Z_3^{(n)} + a^2(Z_3^{(n)})^2, \\ Z_2^{(n+1)} &= a^2(Z_1^{(n)})^2 + 2cZ_1^{(n)}Z_2^{(n)} + 2Z_1^{(n)}Z_3^{(n)} + a^2c^2(Z_2^{(n)})^2 + 2cZ_2^{(n)}Z_3^{(n)} + a^2(Z_3^{(n)})^2, \\ Z_3^{(n+1)} &= a^2(Z_1^{(n)})^2 + 2Z_1^{(n)}Z_2^{(n)} + 2cZ_1^{(n)}Z_3^{(n)} + a^2(Z_2^{(n)})^2 + 2cZ_2^{(n)}Z_3^{(n)} + a^2c^2(Z_3^{(n)})^2. \end{aligned} \tag{7}$$

Letting

$$u_n = \frac{Z_2^{(n)}}{Z_1^{(n)}}, \qquad v_n = \frac{Z_3^{(n)}}{Z_1^{(n)}}$$

from (7) one gets (5) above. Note that if $\bar{\sigma}_n \equiv 1$, then $u_n = v_n$ while they can differ in (5)—so a possibility for symmetry-breaking disappears.

Now let us assume that $J_p \neq 0$ and $\bar{\sigma}_n \equiv 1$. Then the system (6) reduces to a system consisting of five independent variables $Z^{(n)}(1,1,1)$, $Z^{(n)}(2,1,2)$, $Z^{(n)}(1,2,1)$, $Z^{(n)}(2,2,2)$, $Z^{(n)}(3,2,3)$ and with the introduction of new variables

$$\begin{aligned} u_1^{(n)} &= \sqrt{Z^{(n)}(1,1,1)}, \qquad & u_2^{(n)} &= \sqrt{Z^{(n)}(2,1,2)}, \\ u_3^{(n)} &= \sqrt{Z^{(n)}(1,2,1)}, \qquad & u_4^{(n)} &= \sqrt{Z^{(n)}(2,2,2)}, \\ u_5^{(n)} &= \sqrt{Z^{(n)}(3,2,3)}, \end{aligned}$$

straightforward calculations show that

$$\begin{aligned} u_1^{(n+1)} &= ac[b^2{u_1^{(n)}}^2 + 4a^{-1}bu_1^{(n)}u_2^{(n)} + 2(a^2+1){u_2^{(n)}}^2], \\ u_2^{(n+1)} &= a[b^2{u_3^{(n)}}^2 + 2a^{-2}bu_3^{(n)}u_4^{(n)} + 2a^{-2}bu_3^{(n)}u_5^{(n)} + {u_4^{(n)}}^2 + 2a^{-2}u_4^{(n)}u_5^{(n)} + {u_5^{(n)}}^2], \\ u_3^{(n+1)} &= a[{u_1^{(n)}}^2 + 2a^{-1}(b+1)u_1^{(n)}u_2^{(n)} + (a^2b^2+2b+a^2){u_2^{(n)}}^2], \\ u_4^{(n+1)} &= ac[{u_3^{(n)}}^2 + 2a^{-2}bu_3^{(n)}u_4^{(n)} + 2a^{-2}u_3^{(n)}u_5^{(n)} + b^2{u_4^{(n)}}^2 + 2a^{-2}bu_4^{(n)}u_5^{(n)} + {u_5^{(n)}}^2], \\ u_5^{(n+1)} &= a[{u_3^{(n)}}^2 + 2a^{-2}bu_3^{(n)}u_5^{(n)} + 2a^{-2}u_3^{(n)}u_4^{(n)} + {u_4^{(n)}}^2 + 2a^{-2}bu_4^{(n)}u_5^{(n)} + b^2{u_5^{(n)}}^2]. \end{aligned} \tag{8}$$

The total partition function is given in terms of (u_i) by

$$\begin{aligned} Z^{(n)} = {} & [{u_1^{(n)}}^2 + 4a^{-2}u_1^{(n)}u_2^{(n)} + 2(1+a^{-2}){u_2^{(n)}}^2] \\ & + 2[{u_3^{(n)}}^2 + {u_4^{(n)}}^2 + {u_5^{(n)}}^2 + 2a^{-2}(u_3^{(n)}u_4^{(n)} + u_3^{(n)}u_5^{(n)} + u_4^{(n)}u_5^{(n)})]. \end{aligned}$$

We note that, in the paramagnetic phase (high symmetry phase), $u_1 = u_4$ and $au_2 = u_3 = u_5$. For discussing the phase diagram, the following choice of reduced variables is convenient:

$$\begin{aligned} x &= \frac{2au_2 + u_3 + u_5}{u_1 + u_4}, \qquad & y_1 &= \frac{u_1 - u_4}{u_1 + u_4}, \\ y_2 &= \frac{au_2 - u_3}{u_1 + u_4}, \qquad & y_3 &= \frac{au_2 - u_5}{u_1 + u_4}. \end{aligned}$$

The variable x is just a measure of the frustration of the nearest-neighbour bonds and is not an order parameter like y_1, y_2 and y_3. Equations (8) yield:

$$
\begin{aligned}
x' = {} & \frac{1}{2cD}[(a^2b^2 + ab^2 + a^2 + a + 2b + 2a^{-1}b)x^2 + 4(b+1)(a^{-1}+1)x + 4a(a+1) \\
& - 4(b+1)xy_1 + 2(ab^2 + a^2 - 3a^2b^2 - 2b - a)xy_2 \\
& + 2(a^2b^2 - ab^2 - 3a^2 - 2b + a)xy_3 \\
& + 4a(a+1)y_1^2 + (9a^2b^2 + ab^2 + a^2 + 5a - 6b - 2a^{-1}b)y_2^2 \\
& + (a^2b^2 + 5ab^2 + 9a^2 + a - 6b - 2a^{-1}b)y_3^2 \\
& + 4(3b - 1 + 2a^{-1})y_1y_2 + 4(3 - b + 2a^{-1}b)y_1y_3 + 2(6a^{-1}b - ab^2 - 3a^2b^2 \\
& + 10b - 3a^2 - a)y_2y_3 - 8a^2y_1 + 4(1 - 3b + a^{-1}b - a^{-1}b)y_2 \\
& + 4(b - 3 - a^{-1}b + a^{-1})y_3]; \\
y_1' = {} & \frac{2}{D}[2a^{-1}bxy_1 + (a + a^{-1})x(y_2 + y_3) + (a^{-1} - a)(y_2^2 + y_3^2) + 2(a - a^{-1})y_2y_3 \\
& + 2b[aby_1 + a^{-1}(y_2 + y_3)]]; \\
y_2' = {} & \frac{1}{4cD}[(a^2b^2 - ab^2 + a^2 - a + 2b - 2a^{-1}b)x^2 + 4(b+1)(1 - a^{-1})x + 4a(a-1) \\
& - 4(b+1)(a^{-1}+1)xy_1 + 2(a^2 - a - ab^2 - 3a^2b^2 - 2b - 2a^{-1}b)xy_2 \\
& + 2(a^2b^2 - ab^2 - 3a^2 - 2b - 2a^{-1}b - a)xy_3 + 4a(a-1)y_1^2 \\
& + (9a^2b^2 - ab^2 + a^2 - a - 6b - 2a^{-1}b)y_2^2 \\
& + (a^2b^2 - ab^2 + 9a^2 - a - 6b - 2a^{-1}b)y_3^2 \\
& + 4(3b - 1 - a^{-1}b - a^{-1})y_1y_2 + 4(3 - b - a^{-1}b - a^{-1})y_1y_3 \\
& + 2(10b - 2a^{-1}b - ab^2 - 3a^2b^2 - 3a^2 - a)y_2y_3 \\
& - 8a(a+1)y_1 + 4(1 - 3b - a^{-1}b - a^{-1})y_2 + 4(b - 3 - a^{-1}b - a^{-1})y_3]; \\
y_3' = {} & \frac{1}{4cD}[(a^2b^2 + 2b + a^2 - a - ab^2 - 2a^{-1}b)x^2 + 4(b+1)(1 - a^{-1})x + 4a(a-1) \\
& + 4(b+1)(a^{-1} - 1)xy_1 + 2(a^2 + 3a - 3a^2b^2 - ab^2 - 2b + 2a^{-1}b)xy_2 \\
& + 2(a^2b^2 + 3ab^2 - 3a^2 - a - 2b + 2a^{-1}b)xy_3 + 4a(a-1)y_1^2 \\
& + (9a^2b^2 - ab^2 + a^2 - 9a - 6b + 6a^{-1}b)y_2^2 \\
& + (a^2b^2 - 9ab^2 + 9a^2 - a - 6b + 6a^{-1}b)y_3^2 \\
& + 4(3b - 1 + a^{-1}b - 3a^{-1})y_1y_2 + 4(3 - b - 3a^{-1}b + a^{-1})y_1y_3 \\
& + 2(10b - 10a^{-1}b + 3ab^2 - 3a^2b^2 - 3a^2 + 3a)y_2y_3 \\
& - 8a(a-1)y_1 + 4(1 - 3b - a^{-1}b + 3a^{-1})y_2 + 4(b - 3 + 3a^{-1}b - a^{-1})y_3];
\end{aligned} \tag{9}
$$

where

$$
\begin{aligned}
D = {} & (a + a^{-1})x^2 + 4a^{-1}bx + 2ab^2 \\
& + 2ab^2y_1^2 + (3a - a^{-1})(y_2^2 + y_3^2) + 4a^{-1}by_1(y_2 + y_3) + 2(3a^{-1} - a)y_2y_3.
\end{aligned}
$$

If $a = 1$, that is $J_o = 0$, this system of equations reduced to (3).

The average magnetization m for the n-th generation is given by

$$m = 2 - \frac{4[(a^{-2}-1)(y_2-y_3)^2 + 2(y_1 + a^{-2}y_2 + y_3) + (2a^{-2}y_1 + (a^{-2}+1)(y_2+y_3))x]}{B} \tag{10}$$

where

$$\begin{aligned} B &= 3x^2 + 6 + 6y_1^2 + 11(y_2^2 + y_3^2) - 2x(y_2+y_3) - 10y_2y_3 - 4y_1 \\ &\quad + a^{-2}[3x^2 - 5(y_2^2+y_3^2) - 2x(y_2+y_3) + 4(3x - y_2 - y_3 - y_1x + 3y_1(y_2+y_3)) \\ &\quad + 22y_2y_3)]. \end{aligned}$$

If $a = 1$, i.e., $J_o = 0$, then it has more simple form

$$m = 2 - \frac{4(1+x)Y}{3(1+x)^2 - 2(1+x)Y + 3Y^2} \tag{11}$$

where $Y = y_1 + y_2 + y_3$ [16].

The system of four equations finally obtained (9) is essentially complicated than the similar basic equations (1)–(3) of the Ising model [7, 10] and Potts model with $J_o = 0$ [16] respectively. The novelty and complication of the basic equations (9) consist in that it has no explicit fixed points so that we cannot apply method of linearization whereas the basic equations of the Ising model [7, 10] and Potts model with $J_o = 0$ [16] had explicit fixed point and they were investigated using method of linearization. Therefore here the method of linearization is not suitable. Analitic investigation of (9) will be discussed later on. Below we use numerical methods to study its detailed behavior.

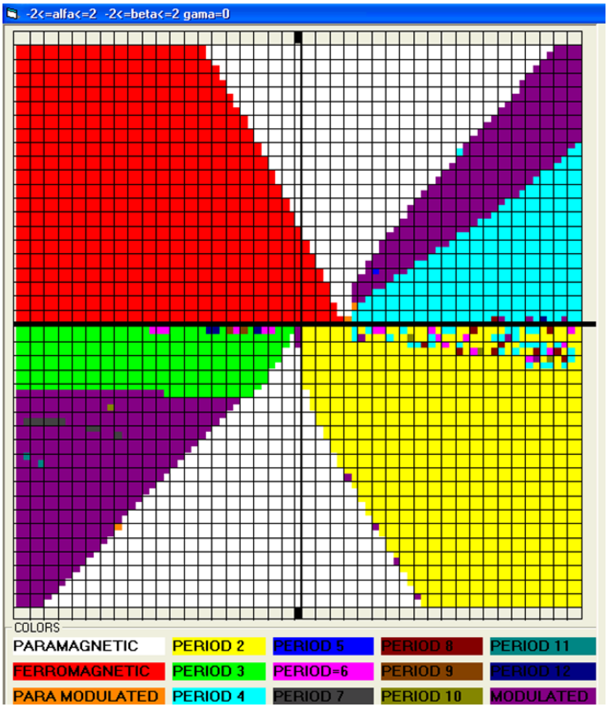
4 Morphology of the Phase Diagram

It is convenient to know the broad features of the phase diagram before discussing the different transitions in more detail. This can be achieved numerically in a straightforward fashion. The recursion relations (9) provide us the numerically exact phase diagram in $(T/J_1, -J_p/J_1, J_o/J_1)$ space. Let $T/J_1 = \alpha$, $-J_p/J_1 = \beta$, $J_o/J_1 = \gamma$ and respectively $c = \exp(\alpha^{-1})$, $b = \exp(-\alpha^{-1}\beta)$ and $a = \exp((2\alpha)^{-1}\gamma)$. Starting from initial conditions

$$x^{(1)} = \frac{2ab^2 + c^2 + 1}{b^2c^3 + c}, \qquad y_1^{(1)} = \frac{b^2c^2 - 1}{b^2c^2 + 1},$$
$$y_2^{(1)} = \frac{ab^2 - c^2}{b^2c^3 + c}, \qquad y_3^{(1)} = \frac{ab^2 - 1}{b^2c^3 + c}$$

that corresponds to boundary condition $\bar{\sigma}_1 \equiv 1$, one iterates the recurrence relations (9) and observes their behavior after a large number of iterations. In the simplest situation a fixed point $(x^*, y_1^*, y_2^*, y_3^*)$ is reached. It corresponds to a paramagnetic phase if $y_1^* = 0$, $y_2^* = 0$, $y_3^* = 0$ or to a ferromagnetic phase if $y_1^*, y_2^*, y_3^* \neq 0$. Note that only if $J_o = 0$ [16] a situation where $y_1^*, y_2^*, y_3^* \neq 0$ but average magnetization (11) $m = 2$ cannot occur (see also [7]). Secondary, the system may be periodic with period p, where case $p = 2$ corresponds to antiferromagnetic phase and case $p = 4$ corresponds to so-called antiphase, that denoted $\langle 2 \rangle$ for compactness. Finally, the system may remain aperiodic. The distinction between a

Fig. 1 (Color online) $\gamma = 0$



truly aperiodic case and one with a very long period is difficult to make numerically. Below we consider periodic phases with period p where $p \leq 12$. All periodic phases with period $p > 12$ and aperiodic phase we will consider as modulated phase.

The resultant phase diagrams for some values of γ are shown in Figs. 1–5.

In Fig. 1 we consider all possible signs of J_p and J_1 with $\gamma = 0$, i.e., $J_o = 0$, whereas in [16] was considered $J_1 > 0$ and $J_p < 0$ only. In first quadrant ($J_p < 0$; $J_1 > 0$) the diagram consists of five phases: ferromagnetic, paramagnetic, modulated, antiphase and paramodulated, all meeting at the multicritical point $(T = 0, -J_p/J_1 = 1/3)$ [16], i.e., this point is five-critical one. In second quadrant ($J_p > 0$; $J_1 > 0$) the diagram consists of ferromagnetic and paramagnetic pases only. Similarly in fourth quadrant ($J_p > 0$; $J_1 < 0$) the diagram consists of antiferromagnetic and modulated phases only. In third quadrant ($J_p > 0$; $J_1 < 0$) phase diagram consists of mainly paramagnetic phase, phase with period 3, modulated phase and a some numbers of islands with different phases. Here these three phases, namely paramagnetic, phase with period 3 and modulated phase, meet at the multicritical point $(T/J_1 = -0.5, J_p/J_1 = 0.5)$, that is three-critical one.

In following figures one can see the significance of parameter J_0. Firstly in Fig. 2 we consider the values of γ that very close to 0, i.e., J_o is rather small.

Looking these phase diagrams one can see that paramagnetic phase in Fig. 1 replaced by ferromagnetic one with preserving other phases, therefore in second quadrant ($J_p > 0$, $J_1 > 0$) the phase diagram consists of ferromagnetic phase only. Above (Fig. 1) we have seen that for $J_o = 0$ a ferromagnetic phase exists for $J_1 > 0$ only and antiferromagnetic phase exists for $J_1 < 0$ only. However at presence one-level interaction $J_o \neq 0$, firstly phase diagram does not contain the paramagnetic phase and secondary phase diagram contains ferromagnetic phase for both $J_1 > 0$ and $J_1 < 0$.

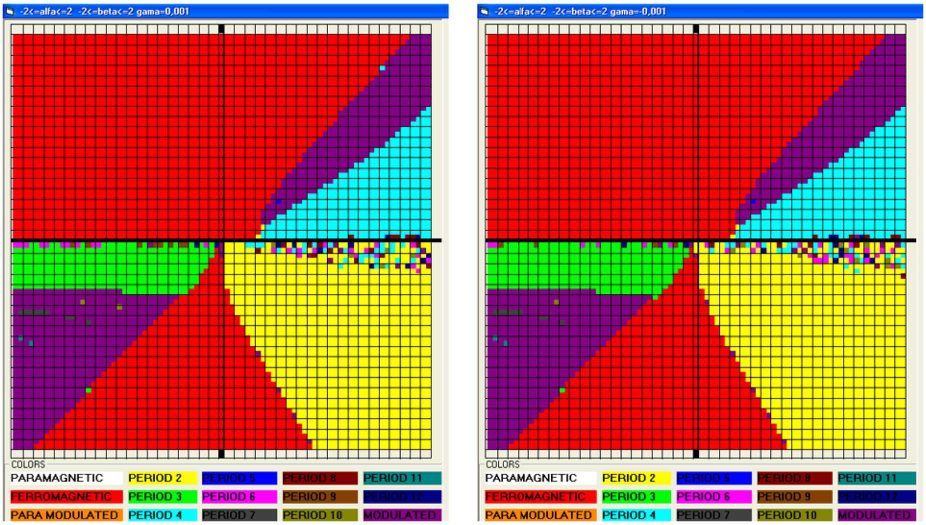


Fig. 2 (Color online) $\gamma = 0.001$ and $\gamma = -0.001$

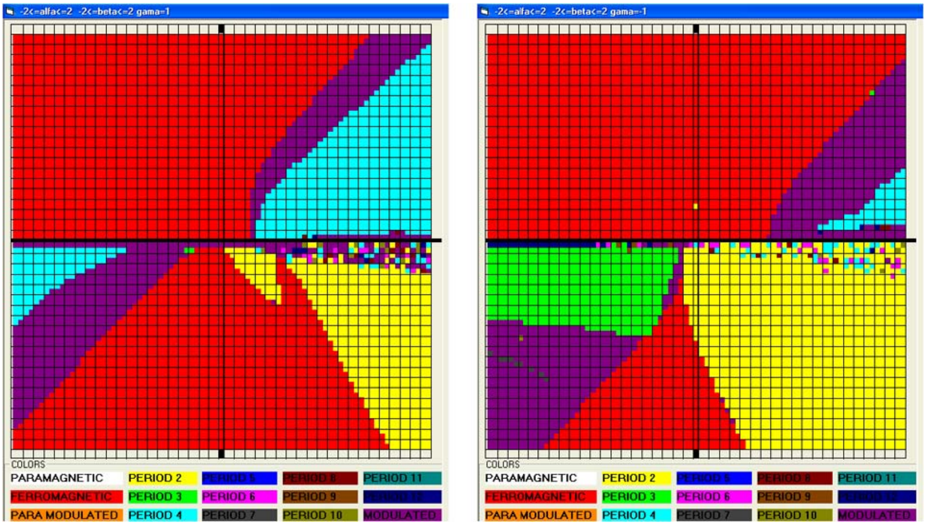


Fig. 3 (Color online) $\gamma = 1$ and $\gamma = -1$

It is easy to explain mathematically the absence of paramagnetic phase for $J_o \neq 0$. In fact if $J_o \neq 0$, then $a \neq 1$ so that the system of recurrent equation (9) does not obtain a fixed point of the form $(x^*, 0, 0, 0)$.

Now in Fig. 3 consider case $|J_1| = |J_o|$, i.e., $|\gamma| = 1$. One can see that for $\gamma = 1$ in first quadrant there is three-critical point $(T = 0, -J_p/J_1 = 0.3)$ and critical points in other quadrants are double-critical ones. For $\gamma = -1$ the critical points in first and fourth quadrant are double-critical and in third quadrant there are two three-critical points $(T/J_1 = -0.1, -J_p/J_1 = -0.1)$ and $(T/J_1 = -0.4, -J_p/J_1 = -1)$. If $|\gamma| = 5$ (Fig. 4)

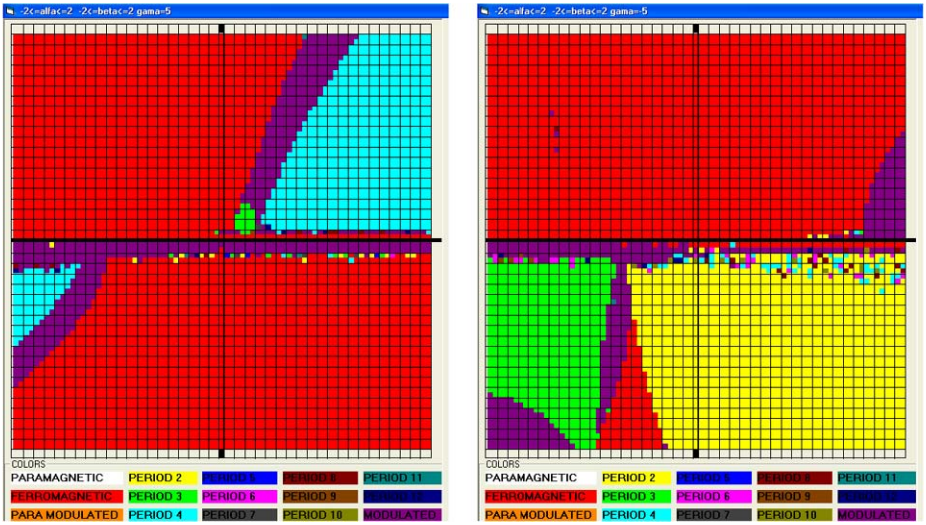


Fig. 4 (Color online) $\gamma = 5$ and $\gamma = -5$

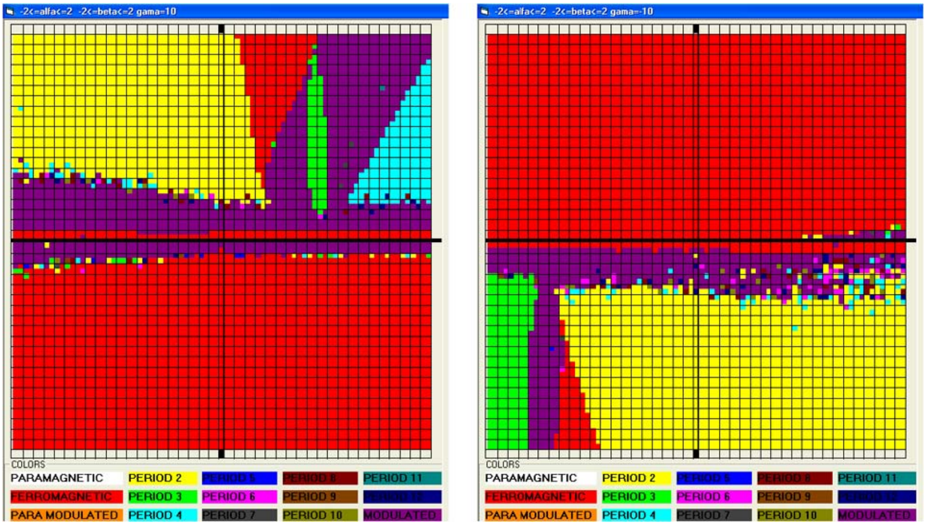


Fig. 5 (Color online) $\gamma = 10$ and $\gamma = -10$

then for $\gamma = 5$ we can find two three-critical points at nonzero temperature in first quadrant, and two double-critical points in third quadrant. For $\gamma = -5$ we have one double-critical point in first quadrant and three three-critical points at nonzero temperature in third quadrant.

Lastly consider the case $|\gamma| = 10$ (Fig. 5). For $\gamma = 10$ we have two three-critical points with higher non zero temperature at them and two double critical points at nonzero temperature also. If $\gamma = -10$ in third quadrant we have three-critical and double critical points at nonzero temperature.

5 Conclusion

The Potts model on a Cayley tree with competing nearest-neighbor interactions J_1, prolonged next-nearest-neighbor interactions J_p and one-level next-nearest-neighbor interactions J_o is rather complicated model than Ising model with competing interactions. The basic equations produced for Potts model with competing prolonged and one-level next-nearest-neighbours and binary nearest neighbours interactions are essentially complete than similar equations produced in [7, 10] and [16]. To investigate them we need to apply some new methods differ than linearization. For $\gamma \neq 0$, firstly the phase diagram does not contain paramagnetic phase and secondary, one can find the multicritical Lifshitz points that are at nonzero temperature where this temperature increase with increasing $|\gamma|$. Although the Cayley tree is not a realistic model, one can hope that the results obtained in the present work can simulate the behavior realistic systems. As pointed by Vannimenus [7], a Cayley tree is a counterpart of the ANNNI model, which is used to provide an approximate description of some materials, such as *CESb* and ferroelectric $NaNO_2$. Other possible realizations of a system which properties similar to those of the Cayley tree are those where there is a gradient in the density of magnetic atoms, as suggested by Moraal [23].

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